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# On a Graph Topology on C(X,Y) with X Compact Hausdorff and Y Tychonoff†

by

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Abstract. We present a characterization of the compact-open topology as a graph topology on the space of continuous functions on a compact Hausdorff space with values in a Tychonoff space.

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## 1. Introduction

Every continuous function from a topological space to a Hausdorff space has a closed graph in the product space. This identification of a continuous function with its graph enables one to view it as an element of the space of closed subsets of the product space and consequently to endow the space of continuous functions C(X,Y) with the relative topology of set-convergence of Choquet [2] and Kuratowski [7]; see [3, pp. 247-254] for details and references. This point of view is hardly novel and has been used to define a topology on the space of upper semicontinuous functions (see [3] and also [1, Theorem 2.78), and also to consider convergence of sequences of maximal monotone operators on a reflexive Banach space (see [1, pp. 351-352 and 360-361] for details and references). However, a characterization of this topology on the space of continuous functions as the compact-open topology of Fox and Arens (see, for example, [4], [8] or [9]) appears not to have been noticed before. This characterization is the main object of this note.

Our result was motivated by a problem in mathematical economics where the space of continuous functions is viewed as a space of economic agents each of whose preferences depend on the distribution of the actions of the others (see [5] and [6] for details).

## 2. The Results

For any topological space $(X, \tau_x)$, let $\mathfrak{F}(X)$ denote the set of closed subsets of X. Let $\tau^K(X)$ denote the topology of set-convergence, also termed the Kuratowski topology, on $\mathfrak{F}(X)$. $\tau^K(X)$ is generated by a sub-base consisting of

$$\{F \in \mathfrak{F}(X): \ F \cap K = \phi\} \text{ and } \{F \in \mathfrak{F}(X): \ F \cap G \neq \phi\}$$

for all compact sets K in X and open sets G in X.

For any two topological spaces $(X, \tau_x)$ and $(Y, \tau_y)$, let $C(X,Y)$ denote the set of continuous functions from X to Y. We shall denote the relativization of $\tau^K(X \times Y)$ to $C(X,Y)$ by $\tau^G(X,Y)$ and shall call this relative topology, the <u>graph topology</u> on $C(X,Y)$.

On $C(X,Y)$ the <u>compact-open topology</u> is denoted by $\tau^C(X,Y)$ and is generated by a sub-base consisting of

$$(K,G) = \{f \in C(X,Y): \ f(K) \subset G\}$$

for all compact sets K in X and open sets G in Y.

Before we state our results, we need an elementary lemma.

<u>Lemma 1</u>. <u>Let</u> $(X,\tau_x)$, $(Y,\tau_y)$, $(Z,\tau_z)$ <u>be topological spaces such that</u> $(Y,\tau_y)$ <u>is a subspace of</u> $(Z, \tau_z)$. <u>Then</u> $C(X,Y) \subset C(X,Z)$.

Lemma 1 allows us to consider the relativization of $\tau^K(X \times Z)$ on $C(X,Y)$. We shall denote this topology by $\tau^G(X,Z)$. We can now state

<u>Theorem</u>. <u>Let</u> $(X,\tau_x)$, $(Z,\tau_z)$ <u>be compact Hausdorff spaces and</u> $(Y,\tau_y)$ <u>a subspace of</u> $(Z,\tau_z)$. <u>Then</u> $\tau^C(X,Y)$ <u>is identical to</u> $\tau^G(X,Z)$ <u>on</u> $C(X,Y)$.

This result has a number of corollaries, the first of which is rather straightforward.

<u>Corollary 1</u>. <u>Let</u> $(X,\tau_x)$ <u>and</u> $(Y,\tau_y)$ <u>be compact Hausdorff spaces. Then</u> $\tau^C(X,Y)$ <u>is identical to</u> $\tau^G(X,Y)$ <u>on</u> $C(X,Y)$.

Our next corollary can be viewed as the principal result of this note. It is phrased in terms of a completely regular $T_1$-space; also called a Tychonoff space.

Corollary 2. *Let $(X,\tau_x)$ be a compact Hausdorff space and $(Y,\tau_y)$ be a Tychonoff space. Then for any two compact Hausdorff spaces $(Z_1,\tau_{z1})$, $(Z_2,\tau_{z2})$ in which $(Y,\tau_y)$ is embedded, the topologies $\tau^G(X,Z_1)$ and $\tau^G(X,Z_2)$ are identical on $C(X,Y)$ and equal to $\tau^C(X,Y)$ on $C(X,Y)$.*

We can now use the characterization of the compact-open topology given in Corollary 2 to deduce the following.

Corollary 3. *Let $(X,\tau_x)$ be a compact Hausdorff space and $(Y,\tau_y)$ be a Tychonoff space. Then $(C(X,Y), \tau^C(X,Y)$ is a Tychonoff space.*

Corollary 3 is, of course, a well-known result even for the case when $(X,\tau)$ is not necessarily compact Hausdorff. However, our proof for the compact case is very different from that presented in Nagata [8, pp. 272-274] or Willard [9, p. 288, Exercise 43B with Theorem 43.7 as a hint].

## 3. Proofs

We begin with a

Proof of Lemma 1.

Pick any $f \in C(X,Y)$ and $G \in \tau_z$. Certainly $G = (G \cap Y) \cup (G-Y)$. Then $f^{-1}(G) = f^{-1}(G \cap Y) \cup f^{-1}(G-Y)$. Since $(G \cap Y) \in \tau_y$ and $f^{-1}(G-Y) = \phi$, we are done. $\square$

Before we present a proof of the theorem, we record two elementary results for which we could find no reference.

Lemma 2. Let $B(\Sigma)$ denote all finite intersections of elements of $\Sigma$. Then a sufficient condition that two sub-bases $\Sigma$, $\Sigma'$ in $X$ lead to an identical topology on $X$ is that

(i) For each $U \varepsilon \Sigma$ and each $x \varepsilon U$, there is a $U' \varepsilon B(\Sigma')$ with $x \varepsilon U' \subset U$;

(ii) for each $U' \varepsilon \Sigma'$ and each $x \varepsilon U'$, there is a $U \varepsilon B(\Sigma)$ with $x \varepsilon U \subset U'$.

Proof. The proof is a simple consequence of [4, III.3.4].

$\square$

Lemma 3. Let $(X,\tau_x)$ be a compact Hausdorff space and $G_i \varepsilon \tau_x$, $G_i \neq \phi$, $i=1, \ldots, n$, such that $X = \bigcup_{i=1}^{n} G_i$. Then there exist compact sets $K_i \subset G_i$, such that $X = \bigcup_{i=1}^{n} K_i$.

Proof. The assertion is trivially true for $n=1$. Suppose it is true for $n-1$. Let $L = \bigcup_{i=1}^{n-1} G_i$. Since $L \cup G_n = X$, $L^c \cap G_n^c = \phi$. Since $(X,\tau_x)$ is compact Hausdorff, it is normal [4, XI.1.2], and hence there exist $0_1$ and $0_2$ in $\tau_x$ such that $0_1 \cap 0_2 = \phi$ and $L^c \subset 0_1$ and $G_n^c \subset 0_2$. Certainly $0_1^c$ and $0_2^c$ are compact and nonempty and such that $0_1^c \subset L$, $0_2^c \subset G_n$ and $0_1^c \cup 0_2^c = X$.

Now endow $0_1^c$ with the relativization of $\tau_x$ and denote it by $\tau_x^0$. Since $0_1^c$ is closed in $(X,\tau_x)$, $(0_1^c, \tau_x^0)$ is a compact Hausdorff space [4, XI.1.4(3)]. Furthermore $(0_1^c \cap G_i) \varepsilon \tau_x^0$. Since

$$0_1^c = \bigcup_{i=1}^{n-1} (0_1^c \cap G_i),$$

the induction hypothesis applies and we can find compact sets $L_i$ in $(0^c, \tau_x)$ such that $0_1^c = \bigcup_{i=1}^{n-1} L_i$. Since $L_i$ are closed in $(X,\tau_x)$, [4, III.7.3], $X = 0_2^c \cup (\bigcup_{i=1}^{n} L_i)$ and we are done.

$\square$

Proof of Theorem. The first point to be noted is that, following [4, III.7.2(1)], the sub-base for $\tau^G(XxZ)$ consists of

$$[K] = \{f \in C(X,Y): \text{ graph } f \cap K = \phi\}$$

$$\langle G\rangle = \{f \in C(X,Y): \text{ graph } f \cap G \neq \phi\}$$

for all compact sets K in X x Z and open sets G in X x Z. We shall now use Lemma 2 to prove the theorem.

We first show (i) and pick (K,G) and $f \in (K,G)$. Since $G \in \tau_y$, there exists $H \in \tau_z$ such that $G = H \cap Y$. Since $f(K) \subset G$ and $f(K) \subset Y$, $f(K) \subset H$. Hence, graph $f \cap (K \times H^c) = \phi$ and since $(Z, \tau_z)$ is compact Hausdorff, $f \in [K \times H^c]$. This also shows that $[K \times H^c] \subset [K,G]$ and we are done.

Next, pick [K] and $f \in [K]$. Since $K^c$ is open in X x Z, there exists an index set I and open sets $X_i$ and $G_i$ in X and Z respectively such that $K^c = \bigcup_{i\in I}(X_i \times G_i)$. Since $f \in [K]$, graph f is contained in $K^c$ and hence in $\bigcup_{i\in I}(X_i \times G_i)$. Since graph f is a compact set in X x Z, there exists a finite integer n such that

$$\text{graph } f \subset \bigcup_{i=1}^{n} (X_i \times G_i).$$

But this can be rewritten as

$$\text{graph } f = \bigcup_{i=1}^{n} (\text{graph } f) \cap (X_i \times G_i).$$

But (graph $f \cap (X_i \times G_i)$) is an open set in graph f and hence we can appeal to Lemma 3 to find sets $L_i$, compact in graph f, such that graph $f = \bigcup_{i=1}^{n} L_i$ and

$$(*) \qquad L_i \subset (\text{graph } f \cap (X_i \times G_i)) \qquad i = 1, \ldots, n.$$

Without loss of generality, let $L_i \neq \phi$ for all $i=1, \ldots, n$. Now let $K_i = \text{proj}_X L_i$. Since $L_i$ are closed in graph f and since graph f is closed in $(X \times Z, \tau_{x \times z})$, and since the projection map is a closed map, $K_i$ is compact in $(X, \tau_x)$. Furthermore, $\bigcup_{i=1}^{n} K_i = X$ and from (*) $K_i \subset X_i$ for all $i=1, \ldots, n$. For any $x \in K_i$, $(x,f(x)) \in L_i \subset (X_i \times G_i)$. Hence $f(K_i) \subset G_i$. Let $H_i = G_i \cap Y$ and observe that $f \in \bigcap_{i=1}^{n} (K_i, H_i)$. Since $H_i \in \tau_y$ and $K_i$ are compact in $(X,\tau_x)$, $(K_i,H_i)$ is a sub-basic set for $\tau^c(X,Y)$. Now pick any $g \in \bigcap_{i=1}^{n} (K_i,H_i)$. For all $i=1, \ldots, n$, $g(K_i) \subset H_i$ and hence graph $g \subset \bigcup_{i=1}^{n} (K_i \times H_i) \subset \bigcup_{i=1}^{n} (X_i \times G_i) \subset K^c$ and hence graph $g \cap K = \phi$. This implies $g \in [K]$. Since g was arbitrary,

$$\bigcap_{i=1}^{n} (K_i,H_i) \subset [K]$$

and we are done.

For our final step, pick $\langle G \rangle$ and $f \in \langle G \rangle$. Since graph $f \cap G \neq \phi$, there exists $x \in X$ such that $(x,f(x)) \in G$. Since $G \in \tau_{xxz}$, there

exists $X_1 \varepsilon \tau_x$ and $H \varepsilon \tau_z$ such that $(X_1 \times H) \subset G$ and $(x,f(x)) \varepsilon X_1 \times H$. Let $H' = H \cap Y$. Then $f \varepsilon (\{x\}, H')$. Since $\{x\}$ is compact in $(X,\tau_x)$ and $H' \varepsilon \tau_y$, $(\{x\},H')$ is a sub-basic set for $\tau^c(X,Y)$.

Now pick any $g \varepsilon (\{x\},H')$. Since $g(x) \subset H' = (H \cap Y)$, $(x,g(x)) \varepsilon (X_1 x H) \subset G$. Hence graph $g \cap G \neq \phi$ and $g \varepsilon \langle G \rangle$. Since g was arbitrary, we have shown

$$[\{x\}, H'] \subset \langle G \rangle.$$

We can now appeal to Lemma 1 to finish the proof of the theorem.

$\square$

Proof of Corollary 1. This is obvious once we choose Y to be equal to Z.

$\square$

Proof of Corollary 2. The fact that this embedding is always possible is guaranteed by [9, Theorem 14.13]. The result now follows directly from the theorem.

$\square$

Proof of Corollary 3. From Corollary 2, we know that $\tau^c(X,Y)$ is identical to $\tau^G(X,Z)$, where $(Z,\tau_z)$ is any compact Hausdorff space in which $(Y,\tau_y)$ is embedded. Now $\tau^G(X,Z)$ is the relativization of $\tau^K(XxZ)$. Since XxZ is a compact Hausdorff space, $(XxZ, \tau^K(XxZ))$, is a compact Hausdorff space [1, Theorem 2.76]. Hence $\tau^K(XxZ)$ is normal and therefore Tychonoff. Since every subspace of a Tychonoff space is Tychonoff [9, Theorem 14.10(a)], we are done.

$\square$

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